HURWITZ THEOREM:

Statement: If the functions $f_n(z)$ are analytic and $\neq 0$ in a region Ω and if $f_n(z)$ converges to $f(z)$ uniformly on every compact subset of Ω, then $f(z)$ is either identically zero or never equal to zero in Ω.

Proof:

Given that $f_n(z)$ are analytic and $\neq 0$ in a region Ω.

Also given that $f_n(z)$ converges to $f(z)$ uniformly on every compact subset of Ω.

Therefore, by theorem 1 $f(z)$ is analytic in Ω.

Claim: $f(z) \equiv 0 \ \forall \ z \in \Omega$ or $f(z) \neq 0 \ \forall \ z \in \Omega$.

Suppose $f(z) \equiv 0 \ \forall \ z \in \Omega$, then the theorem is true.

Suppose not , then we have to prove $f(z) \neq 0 \ \forall \ z \in \Omega$.

Let $z_0 \in \Omega$. Then there exists $r > 0$ such that $f(z) \neq 0$ on $0 < |z - z_0| \leq r$ and $|f(z)|$ has a positive minimum of the circle C, $|z - z_0| = r$.

Thus $\frac{1}{f(z)}$ has a maximum on C.

Now $\frac{1}{f_n(z)} \rightarrow \frac{1}{f(z)}$ uniformly on C, since $f_n(z) \rightarrow f(z) uniformly\ on\ C.$

Also $f'_n(z) \rightarrow f'(z) uniformly\ on\ C.$

Therefore $\frac{f_n'(z)}{f_n(z)} \rightarrow \frac{f'(z)}{f(z)}$ on C.

That is, $\lim\limits_{n\to\infty} \frac{1}{2\pi i}\int_C \frac{f_n'(z)}{f_n(z)}dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz.$

We know that $\frac{1}{2\pi i}\int_C \frac{f_n'(z)}{f_n(z)}dz$ is the number of zeros of $f_n(z)$ enclosed by C.

Since $f_n(z) \neq 0$ on Ω, we get $f_n(z) \neq 0$ on C.

Hence $\lim\limits_{n\to\infty} \frac{1}{2\pi i}\int_C \frac{f_n'(z)}{f_n(z)}dz = 0.$

This implies $\int_C \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = 0.$

i.e, The number of zeros of f(z) enclosed by C = 0.

This implies $f(z_0) \neq 0$. Since z_0 is arbitrary, f(z) ≠ 0 for all z in Ω

Taylor series:

Statement:

If $f(z)$ is analytic in the region Ω, containing z_0, then the representation

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

Is valid in the largest open disk of center z_0 contained in Ω.

Proof: By taylor's theorem,

WKT $f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + f_{n+1}(z)(z - z_0)^{n+1}$

Where $f_{n+1}(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)d\xi}{(\xi - z_0)^{n+1}(\xi - z)}$. Here C is the circle $|\xi - z_0| \leq r$ contained in Ω and z lies inside C.

Let M denote the maximum of $|f(\xi)|$ on C.

Then $|f_{n+1}(z)(z - z_0)^{n+1}| \leq \frac{1}{2\pi}\int_C \frac{|f(\xi)||d\xi|}{|\xi - z_0|^{n+1}|\xi - z|}|z - z_0|^{n+1}$

$$\leq \frac{1}{2\pi}M|z - z_0|^{n+1}\int_C \frac{|d\xi|}{|\xi - z_0|^{n+1}(|\xi - z_0 + z_0 - z|)}$$

$$\leq \frac{M|z - z_0|^{n+1}}{2\pi\, r^{n+1}(r - |z - z_0|)}\int_C |d\xi|$$

$$= \frac{M|z - z_0|^{n+1}2\pi r}{2\pi\ \ r^{n+1}\ (r - |z - z_0|)}$$

$$= \frac{M|z - z_0|^{n+1}}{r^n(r - |z - z_0|)}$$

For z lying in a closed disk $|z - z_0| \leq \rho < r$, we have

$|f_{n+1}(z)(z - z_0)^{n+1}| \leq \frac{M\,\rho^{n+1}}{r^n(r - \rho)}$

$$= \frac{M\rho}{r - \rho}\left(\frac{\rho}{r}\right)^n \to 0\ as\ n \to \infty$$

Hence $f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$

Mittag- Leffler's theorem:

Let $\{b_\gamma\}$ be a sequence of complex numbers with $\lim_{\gamma \to \infty} b_\gamma = \infty$ and let $P_\gamma(\xi)$ be polynomial without constant term. Then there are functions which are meromorphic in the whole plane with poles at the points b_γ and the corresponding singular part $P_\gamma(\frac{1}{z - b_\gamma})$. Moreover the most general meromorphic function of this kind can be written as

$f(z) = \sum_\gamma \left[P_\gamma\left(\frac{1}{z - b_\gamma}\right) - P_\gamma(z)\right] + g(z)$ where $P_\gamma(z)$ are suitably chosen polynomials and g(z) is analytic in the whole plane.

Proof: The function $P_\gamma\left(\frac{1}{z - b_\gamma}\right)$ is analytic in $|z| < |b_\gamma|$.

This implies $P_\gamma\left(\frac{1}{z - b_\gamma}\right)$ has a taylor series expansion about the point zero.

Let $p_\gamma(z)$ be the partial sum of the series ending with the term of degree n_γ.

Let M_γ be the maximum value of $P_\gamma(\xi)$ on $|\xi| \le |\frac{b_\gamma}{2}|$, then for $|z| \le |\frac{b_\gamma}{4}|$

$$\left|P_\gamma\left(\frac{1}{z-b_\gamma}\right) - p_\gamma(z)\right| = \left|p_{\gamma+1}(z)\ \ (z)^{n_\gamma+1}\right|$$

$$= \left|(z)^{n_\gamma+1}\frac{1}{2\pi i}\int_{|\xi|=\left|\frac{b_\gamma}{2}\right|}\frac{P_\gamma(\xi)d\xi}{\xi^{n_\gamma+1}(\xi-z)}\right|$$

$$\le |z|^{n_\gamma+1}\frac{1}{2\pi}\frac{M_\gamma}{\left(\frac{b_\gamma}{2}\right)^{n_\gamma+1}}\frac{2\pi\left|\frac{b_\gamma}{2}\right|}{\left|\frac{b_\gamma}{4}\right|}$$

$$= 2M_\gamma\left(\frac{2|z|}{|b_\gamma|}\right)^{n_\gamma+1}$$

$$\le \left(\frac{1}{2^2}\right)^{n_\gamma+1} M_\gamma 2^{n_\gamma+2}$$

Now choose n_γ such that $2^{n_\gamma} \ge M_\gamma\ 2^\gamma$.

Then $\left|P_\gamma\left(\frac{1}{z-b_\gamma}\right) - p_\gamma(z)\right| \le \left(\frac{1}{2^2}\right)^{n_\gamma+1}\ \frac{2^{n_\gamma}}{2^\gamma}2^{n_\gamma+2} = \frac{1}{2^\gamma}$

But $\sum\frac{1}{2^\gamma}$ converges and so by comparison test, $\sum_\gamma\left[P_\gamma\left(\frac{1}{z-b_\gamma}\right) - p_\gamma(z)\right]$ converges to a function h(z)

Hence h(z) is a meromorphic function with singular part $P_\gamma\left(\frac{1}{z-b_\gamma}\right)$.

Let f(z) be a meromorphic function with poles b_γ.

Then g(z)=f(z)-h(z) is an analytic function in the whole plane.

That is f(z) =h(z)+g(z) = $\sum_\gamma\left[P_\gamma\left(\frac{1}{z-b_\gamma}\right) - p_\gamma(z)\right] + g(z)$

EXAMPLE 1: $f(z) = \frac{e^{2z}}{(z-1)^3}$

About z=1, $f(z) = \frac{e^{2(z-1)+2}}{(z-1)^3}$

$$= \frac{e^2e^{2(z-1)}}{(z-1)^3}$$

$$\therefore f(z) = e^2\frac{1}{(z-1)^3}\left[1+\frac{2(z-1)}{1!}+\frac{4(z-1)^2}{2!}+\cdots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2 1!}+\frac{4}{(z-1)2!}+\frac{8}{3!}+\frac{16(z-1)}{4!}\ldots\right]$$

$$= P_\gamma\left(\frac{1}{z-1}\right) + \ g(z)$$